ON EQUILIBRIUM STABILITY OF AN ELASTIC – VISCOPLASTIC MEDIUM

A. N. Sporykhin

We consider the stability of an elastic–plastic medium when one part of the body is in an elastic state and the other part in a plastic state.

The results obtained in [1] are generalized for the stability of a deformation of an elastic–viscoplastic hardening medium, assuming that the whole body is in a plastic state.

Linearized relations are applied in studying the stability of a thick-walled tube subject to a planar deformation under the action of an internal pressure for various cases of loading behavior, both "following" as well as "dead zone" type, for small deviations of the body from an unperturbed equilibrium.

An analogous problem was considered in [2] for a quasistatic formulation for a tube of an ideal plastic hardening material, also in accord with the theory of small elastic–plastic deformations.

1. We consider the unperturbed equilibrium of a hardening elastic–plastic body of volume V, characterized by a vector of displacements $u_i^\circ(x_k, t)$, a stress tensor $\sigma_{ij}^\circ(x_k, t)$, and volume and surface force vectors F_i° and p_i°, and we let $x_i(\zeta, t)$ be the surface separating the domains of elastic and plastic behavior of the medium.

The study of the stability of equilibrium of a body of volume V reduces to the solution of variational equations and corresponding boundary conditions [3], which, for the case considered here, have the form

$$(\overset{+}{\sigma}_{ij} + \overset{\circ}{\sigma}_{jk} u_{i,\overset{+}{k}})_{,j} + F_i^+ - \rho \ddot{u}_i^{+} = 0, \qquad (\overset{+}{\sigma}_{ij} + \overset{\circ}{\sigma}_{jk} u_{i,\overset{+}{k}}) n_j = \dot{p}_i^{+} \tag{1.1}$$

Components characteristic of the perturbed motion are denoted with a plus sign.

On the elastic–plastic boundary the stresses and displacements are continuous. From this it follows that

$$[\overset{+}{\sigma}_{ij} + \overset{\circ}{\sigma}_{ij,k} x_k^{+}] \nu_j = 0, \quad [u_i^{+} + \overset{\circ}{u}_{i,k} x_k^{+}] = 0 \quad (i, j = 1, 2) \tag{1.2}$$

Here the square brackets denote the difference of corresponding quantities.

The defining variational relations may be written in the following forms [1]:

a) in the plastic zone

$$\begin{gathered}
[2\mu \overset{+}{\sigma}_{ij} - {}^2/_3 \mu (3\lambda + 2\mu) \overset{+}{e}_{kk} \delta_{ij} - c (\lambda \overset{+}{e}_{kk} \delta_{ij} + 2\mu \overset{+}{e}_{ij} - \overset{+}{\sigma}_{ij}) \\
- \eta (\lambda \dot{\overset{+}{e}}_{kk} \delta_{ij} + 2\mu \dot{\overset{+}{e}}_{ij} - \dot{\overset{+}{\sigma}}_{ij})] (\overset{\circ}{s}_{ij} - c e_{ij}^{p\circ} - \eta \dot{e}_{ij}^{p\circ}) = 0 \\
(1 + \eta \psi_0)(\lambda \dot{\overset{+}{e}}_{kk} \delta_{ij} + 2\mu \dot{\overset{+}{e}}_{ij} - \dot{\overset{+}{\sigma}}_{ij}) + c \psi_0 (\lambda \overset{+}{e}_{kk} \delta_{ij} + 2\mu \overset{+}{e}_{ij} - \overset{+}{\sigma}_{ij}) \\
= k^{-2} (\lambda \dot{\overset{+}{e}}_{nn} \delta_{kl} + 2\mu \dot{\overset{+}{e}}_{kl} - \dot{\overset{+}{\sigma}}_{kl}) (\overset{\circ}{s}_{kl} - c e_{kl}^{p\circ} - \eta \dot{e}_{kl}^{p\circ}) (\overset{\circ}{s}_{ij} - c e_{ij}^{p\circ} - \eta \dot{e}_{ij}^{p\circ}) \\
+ \psi_0 (2\mu \sigma_{ij}^{+} - {}^2/_3 \mu (3\lambda + 2\mu) e_{kk}^{+} \delta_{ij})
\end{gathered} \tag{1.3}$$

b) in the elastic zone

$$\overset{+}{\sigma}_{ij} = \lambda \overset{+}{e}_{kk} \delta_{ij} + 2\mu \overset{+}{e}_{ij} \tag{1.4}$$

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Here the deformations are connected with the displacements through the formulas

$$e_{ij} = {}^1/_2(u_{i,j}^{+} + u_{j,i}^{+} + \overset{\circ}{u}_{k,i} u_{k,j}^{+} + \overset{\circ}{u}_{k,j} u_{k,i}^{+}) \tag{1.5}$$

Applying the method developed in Section 2 of [1], we can, in an analogous way, reduce the boundary problems (1.1)-(1.5) to the study of a system of differential equations with constant coefficients.

Moreover, the equations of equilibrium and the boundary conditions on the surface (1.1) reduce to the equations

$$(\textstyle\sum_{ij} + \overset{\circ}{\sigma}_{jk} U_{i,k})_{,j} + F_i + \rho\omega^2 U_i = 0, \quad (\textstyle\sum_{ij} + \overset{\circ}{\sigma}_{jk} U_{i,k}) n_j = p_i \tag{1.6}$$

Hereafter we use the index p for quantities relating to the plastic domain and the index e for those relating to the elastic domain.

From Eqs. (1.3) we obtain

$$\begin{gathered}\textstyle\sum_{ij}^{p} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} - \frac{4\mu^2}{k^2(2\mu + c + s\eta)}(E_{kl} - {}^1/_3 E_{mm}\delta_{kl})(\overset{\circ}{s}_{kl} - ce_{kl}^{p^\circ}) \\ \times (\overset{\circ}{s}_{ij} - ce_{ij}^{p^\circ}), \quad s = i\omega\end{gathered} \tag{1.7}$$

In the elastic domain we have the relations

$$\textstyle\sum_{ij}^{e} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} \tag{1.8}$$

Moreover, from formulas (1.5) we obtain

$$E_{ij} = {}^1/_2(U_{i,j} + U_{j,i} + \overset{\circ}{u}_{k,i} U_{k,j} + \overset{\circ}{u}_{k,j} U_{k,i}) \tag{1.9}$$

Conditions (1.2) assume the form

$$[\textstyle\sum_{ij} + \overset{\circ}{\sigma}_{ij,k} X_k] = 0, \qquad [U_i + \overset{\circ}{u}_{i,k} X_k] = 0 \tag{1.10}$$

2. Consider a tube with radii r_1 and r_2, subject to the action of an internal pressure p.

It is well known [3] that the character of the loading behavior for small departures of the body from unperturbed equilibrium may have an essential influence on stability (instability).

In contrast to [2], where it was assumed that as the result of small perturbations the load does not change its direction, we consider the case of a following load. In this case the right member of the second of Eqs. (1.6) has the form

$$p_i = p_j^{\circ} U_{i,j} \tag{2.1}$$

The loaded and deformed state of the tube, made of a hardening elastic–viscoplastic material, in the case of a plane deformation, is determined, up to the point of stability loss, by the expressions

$$\begin{gathered}\sigma_r^{p^\circ} = -p_0 + (2 + c_0)^{-1}[4k_0 \ln(r/\alpha) - c_0 C(r^{-2} - \alpha^{-2})] \\ \sigma_\theta^{p^\circ} = -p_0 + (2 + c_0)^{-1}[4k_0(1 + \ln(r/\alpha)) + c_0 C(r^{-2} + \alpha^{-2})], \quad \sigma_{r\theta}^{p^\circ} = 0 \\ e_r^{p^\circ} = (k_0 r^2 - C)/r^2(2 + c_0), \quad e_\theta^{p^\circ} = -(k_0 r^2 - C)/r^2(2 + c_0), \quad e_{r\theta}^{p^\circ} = 0 \\ \sigma_r^{e^\circ} = C(1 - r^{-2}), \quad \sigma_\theta^{e^\circ} = C(1 + r^{-2}), \quad \sigma_{r\theta}^{e^\circ} = 0, \quad u^\circ = C/r, \quad C = k_0\gamma^2, \\ \alpha = r_1 | r_2\end{gathered} \tag{2.2}$$

where r is the dimensionless flow radius. Here and in the sequel, all quantities having the dimension of length are referred to the external radius r_2; those having the dimension of stress are referred to the shear modulus μ and denoted with a zero subscript.

The radius of the elastic–plastic boundary γ satisfies the equation

$$\gamma^2(1 - c_0/(2 + c_0)\alpha^2) = 4\ln(\gamma/\alpha)/(2 + c_0) - p_0/k_0 + 1 \tag{2.3}$$

We remark that the pressure at which the whole tube enters the plastic state is defined by the expression

$$P = -(2 + c_0)^{-1}(4\ln\alpha - c_0/\alpha^2)$$

The equilibrium equations (1.6) for the perturbation components (1.7), (1.8) may be represented, in the case of a plane form for the stability loss, in the form

$$\sum_{r,r} + r^{-1}\sum_{r\theta,\theta} + r^{-1}(\sum_r - \sum_\theta) + (\sigma_r^\circ U_{r,r})_{,r}$$
$$+ r^{-1}(r^{-1}U_{r,\theta\theta} - 2r^{-1}U_{\theta,\theta} + U_{r,r} - r^{-1}U_r)\sigma_\theta^\circ + \rho_0\omega^2 U_r = 0$$
$$\sum_{r\theta,r} + r^{-1}\sum_{\theta,\theta} + 2r^{-1}\sum_{r\theta} + (\sigma_r^\circ U_{\theta,r})_{,r} + r^{-1}(r^{-1}U_{\theta,\theta\theta} + 2r^{-1}U_{r,\theta} + U_{\theta,r}) \tag{2.4}$$
$$+ \rho_0\omega^2 U_\theta = 0, \quad \rho_0 = \rho r_2^2/\mu$$

From the condition (1.6) we obtain, taking note of Eq. (2.1),

$$\sum_r = 0, \quad \sum_{r\theta} = 0 \quad \text{for} \quad r = 1,\ r = \alpha \tag{2.5}$$

Conditions on the elastic–plastic boundary (1.10) are

$$[U_r] = 0, \quad [U_\theta] = 0, \quad [\sum_r] = 0, \quad [\sum_{r\theta}] = 0 \quad \text{for} \quad r = \gamma \tag{2.6}$$

For an incompressible material the relations (1.7), for the case considered, assume the form

$$S_r^p = 2U_{r,r}^p + a_0[r^{-1}(U_r^p + U_{\theta,\theta}^p) - U_{r,r}^p]$$
$$S_\theta^p = 2r^{-1}(U_r^p + U_{\theta,\theta}^p) - a_0[r^{-1}(U_r^p + U_{\theta,\theta}^p) - U_{r,r}^p] \tag{2.7}$$
$$\sum_{r,\theta}^p = r^{-1}U_{r,\theta}^p + U_{\theta,r}^p - r^{-1}U_\theta^p, \quad a_0 = 4/(2 + c_0 + i\omega\tau)$$

Correspondingly, from Eq. (1.8) for the elastic region $\gamma \le r \le 1$, we obtain

$$S_r^e = 2U_{r,r}^e, \quad S_\theta^e = 2r^{-1}(U_r^e + U_{\theta,\theta}^e), \quad \sum_{r,\theta}^e = r^{-1}U_{r,\theta}^e + U_{\theta,r}^e - r^{-1}U_\theta^e \tag{2.8}$$

We seek the solution in the region $\alpha \le r \le \gamma$ in the form

$$U_r^p = \varphi_1(r)\cos m\theta, \quad U_\theta^p = \varphi_2(r)\sin m\theta \tag{2.9}$$

After substituting the relations (2.9) into the equilibrium equations (1.6), noting thereby the relationship (2.7) between the stresses and deformations and also the condition of incompressibility, we obtain a differential equation for the function $\varphi_1(r)$

$$r^4(1 + \sigma_r^{p\circ})\varphi_1^{(IV)} + r^3[6 + \sigma_\theta^{p\circ} + 5\sigma_r^{p\circ} + 2r\sigma_{r,r}^{p\circ}]\varphi_1^{(III)}$$
$$+ r^2[5 - 2m^2(1 - 2a_0) + (3 - m^2)(\sigma_r^{p\circ} + \sigma_\theta^{p\circ}) + r(r\sigma_{r,rr}^{p\circ}$$
$$+ 7\sigma_{r,r}^{p\circ} + r\rho_0\omega^2)]\varphi_1^{(II)} + r[-1 - 2m^2(1 + \sigma_\theta^{p\circ}) \tag{2.10}$$
$$+ r(2 - m^2)\sigma_{r,r}^{p\circ} + 2r^2\sigma_{r,rr}^{p\circ} + 3r^2\rho_0\omega^2]\varphi_1^{(I)}$$
$$+ [1 - 2m^2 + m^4 + m^2(m^2 - 2)\sigma_\theta^{p\circ} + r^2(1 - m^2)\rho_0\omega^2]\varphi_1 = 0$$

Similarly we seek a solution in the region $\gamma \le r \le 1$ in the form

$$U_r^e = f_1(r)\cos m\theta, \quad U_\theta^e = f_2(r)\sin m\theta \tag{2.11}$$

In a manner analogous to that for the Eq. (2.10), we obtain for the function $f_1(r)$ the equation

$$r^4(1 + \sigma_r^{e\circ})f_1^{(IV)} + r^3[6 + \sigma_\theta^{e\circ} + 5\sigma_r^{e\circ} + 2r\sigma_{r,r}^{e\circ}]f_1^{(''')}$$
$$+ r^2[5 - 2m^2 + (3 - m^2)(\sigma_r^{e\circ} + \sigma_\theta^{e\circ}) + r(r\rho_0\omega^2 + 7\sigma_{r,r}^{e\circ} + r\sigma_{r,rr}^{e\circ})]f_1^{('')} \tag{2.12}$$
$$+ r[-1 - 2m^2(1 + \sigma_\theta^{e\circ}) + 3r^2\rho_0\omega^2 + r(2 - m^2)\sigma_{r,r}^{e\circ} + 2r^2\sigma_{r,rr}^{e\circ}]f_1^{(')}$$
$$+ [1 + m^2(m^2 - 2)(1 - \sigma_\theta^{e\circ}) + r^2(1 - m^2)\rho_0\omega^2]f_1 = 0$$

In Eqs. (2.10) and (2.12) the stresses σ°_r, σ°_θ and their derivatives are defined by the formulas (2.2).

If in the equilibrium Eqs. (1.6) we assume the terms containing the external load to be small, i.e., if we neglect the difference between the geometry of the initial unperturbed state, whose stability is in question, and the geometry of the other states close to it, and also if we assume that m = 1, which corresponds to the first critical force, we obtain from Eqs. (2.10) and (2.12) the simplified equations

$$r^3\varphi_1^{(IV)} + 6r^2\varphi_1^{(''')} + r(3 + 4a_0 + \rho_0\omega^2 r^2)\varphi_1^{('')} + (-3 + 4a_0 + 3\rho_0\omega^2 r^2)\varphi_1^{(')} = 0 \tag{2.13}$$
$$r^3 f_1^{(IV)} + 6r^2 f_1^{(''')} + r(3 + \rho_0\omega^2 r^2)f_1^{('')} + (-3 + 3\rho_0\omega^2 r^2)f_1^{(')} = 0 \tag{2.14}$$

From Eqs. (2.4), (2.7), (2.8), (2.9), and (2.11), noting that the tube material is incompressible, we obtain, for the case under consideration, the following relationships:

a) in the plastic zone

$$\begin{gathered}\textstyle\sum_r^p = -[r^2\varphi_1^{(''')} + 4r\varphi_1^{('')} + (-2 + 4a_0 + \rho_0\omega^2 r^2)\varphi_1^{(')} + \rho_0\omega^2 r\varphi_1)\cos\theta + K^p \\ \textstyle\sum_\theta^p = -[r^2\varphi_1^{(''')} + 4r\varphi_1^{('')} + (2 + \rho_0\omega^2 r^2)\varphi_1^{(')} + \rho_0\omega^2 r\varphi_1]\cos\theta + K^p \\ \textstyle\sum_{r\theta}^p = -(r\varphi_1^{('')} + \varphi_1^{(')})\sin\theta\end{gathered} \tag{2.15}$$

b) in the elastic zone

$$\begin{gathered}\textstyle\sum_r^e = -[r^2 f_1^{(''')} + 4r f_1^{('')} + (-2 + \rho_0\omega^2 r^2) f_1^{(')} + \rho_0\omega^2 r f_1]\cos\theta + K^e \\ \textstyle\sum_\theta^e = -[r^2 f_1^{(''')} + 4r f_1^{('')} + (2 + \rho_0\omega^2 r^2) f_1^{(')} + \rho_0\omega^2 r f_1]\cos\theta + K^e \\ \textstyle\sum_{r\theta}^e = -(r f_1^{('')} + f_1^{(')})\sin\theta\end{gathered} \tag{2.16}$$

where K^p and K^e are determined from the boundary conditions

$$\begin{gathered}\cos\theta\,[\alpha^2\varphi_1^{(''')} + \alpha\,(6 - 4a_0 - \rho_0\omega^2\alpha^2)\,\varphi_1^{('')} + \rho_0\omega^2\alpha\varphi_1] = K^{(p)} \text{ for } r = \alpha \\ \cos\theta\,[f_1^{(''')} + (6 - \rho_0\omega^2) f_1^{('')} + \rho_0\omega^2 f_1] = K^e \text{ for } r = 1\end{gathered} \tag{2.17}$$

The boundary conditions (2.5) together with relations (2.15) and (2.16) yield

$$\varphi_1^{(')} + \alpha\varphi_1^{('')} = 0 \quad \text{for } r = \alpha, \quad f_1^{(')} + f_1^{('')} = 0 \quad \text{for } r = 1 \tag{2.18}$$

From relations (2.6), (2.15), and (2.16) we find that for $r = \gamma$

$$\begin{gathered}f_1 = \varphi_1, \qquad f_1^{(')} = \varphi_1^{(')}, \qquad f_1^{('')} = \varphi_1^{('')} \\ \cos\theta\,[\gamma^2(\varphi_1^{(''')} - f_1^{(''')}) + 4a_0\varphi_1^{(')}] + K^e - K^p = 0\end{gathered} \tag{2.19}$$

The solution of Eqs. (2.13) and (2.14) may be found in the form

$$\begin{gathered}\varphi_1 = \Big[\ln r + \sum_{n=1}^{\infty}\frac{1}{2n}M_0 r^{2n}\Big]A_1 + \Big[\frac{r^k}{k} + \sum_{n=1}^{\infty}\frac{r^{2n+k}}{2n+k}M_{-k}\Big]A_2 \\ + \Big[\frac{r^{-k}}{k} + \sum_{n=1}^{\infty}\frac{r^{2n-k}}{2n-k}M_k\Big]A_3 + A_4 \\ f_1 = (rb)^{-1}\,[I_3(r\sqrt{b}) - 4(r\sqrt{b})^{-1} I_2(r\sqrt{b})]\,C_1 \\ + (rb)^{-1}\,[Y_3(r\sqrt{b}) - 4(r\sqrt{b})^{-1} Y_2(r\sqrt{b})]\,C_2 + r^{-2}C_3 + C_4\end{gathered} \tag{2.20}$$

$$\begin{gathered}M_k = \frac{(-1)^n b^n (n - k/2)!}{\prod_{l=1}^{n}[(al - 3 - ak/2) + (2l - 1 - k)(l - 1 - k/2)(2l + 3 - k)]}, \qquad b = \rho_0\omega^2 \\ k = 2\sqrt{1 - a_0}, \quad a = 3 + 4a_0, \quad M_k = M_{-k} \text{ for } k = -k, \quad M_k = M_0 \text{ for } k = 0\end{gathered} \tag{2.21}$$

Here I_ν, Y_ν, ($\nu = 2,3$) are Bessel functions of orders one and two, and C_i and A_i ($i = 1, \ldots, 4$) are constants of integration.

If we substitute the quantities (2.20) and (2.21) into the relations (2.18) and (2.19), we obtain a linear homogeneous system of algebraic equations in six arbitrary constants. Since in the case involving a loss of stability this system must have a nonzero solution, its determinant must vanish, i.e.,

$$|a_{ik}| = 0 \quad (i,k = 1,\ 2,\ \ldots,\ 6) \tag{2.22}$$

Here

$$\begin{gathered}a_{11} = \frac{1 + \sqrt{b}}{\sqrt{b}} I_2(\sqrt{b}) - I_3(\sqrt{b}), \qquad a_{21} = \frac{1}{\gamma}\Big[I_3(\gamma\sqrt{b}) - \frac{4}{\gamma\sqrt{b}} I_2(\gamma\sqrt{b})\Big] \\ a_{31} = 0, \qquad a_{41} = I_2(\gamma\sqrt{b}), \qquad a_{51} = \frac{1}{\sqrt{b}} I_2(\gamma\sqrt{b}) - \gamma I_3(\gamma\sqrt{b}) \\ a_{61} = \gamma[(b-3)I_3(\sqrt{b}) + (b - 2 - 4\sqrt{b})\,I_2(\sqrt{b})] - \gamma^2\Big[I_3(\gamma\sqrt{b}) - \frac{4}{\gamma\sqrt{b}} I_2(\gamma\sqrt{b})\Big] \\ a_{13} = 4, \quad a_{23} = \gamma^{-2}, \quad a_{33} = 0, \quad a_{43} = -2\gamma^{-2}, \quad a_{53} = 6\gamma^{-2}, \quad a_{63} = \gamma(4 - b) + 24\gamma^{-2} \\ a_{16} = 0, \qquad a_{26} = -\Big[-\frac{1}{k}\gamma^{-k} + \sum_{n=1}^{\infty}\frac{1}{2n-k}M_k\gamma^{2n-k}\Big]\end{gathered} \tag{2.23}$$

$$a_{36} = -\frac{k}{\alpha^k} + \sum_{n=1}^{\infty} (2n-k) M_k \alpha^{2n-k}, \qquad a_{46} = -\left[\gamma^{-k} + \sum_{n=1}^{\infty} M_k \gamma^{2n-k}\right]$$

$$a_{56} = \frac{k+1}{\gamma^k} - \sum_{n=1}^{\infty} (2n-k-1) M_k \gamma^{2n-k}, \qquad a_{66} = -\frac{\gamma}{\alpha}\Big\{\Big[(k+1)(k+2)$$

$$-4(k+1) + (-2 + 4a_0 + b\alpha^2) - \frac{b\alpha^2}{k}\Big]\frac{1}{\alpha^k} + \sum_{n=1}^{\infty} R_k M_k \alpha^{2n-k}\Big\}$$

$$+ 4a_0\left[\gamma^{-k} + \sum_{n=1}^{\infty} M_k \gamma^{2n-k}\right] + (k+1)(k+2)\gamma^{-k}$$

$$+ \sum_{n=1}^{\infty} (2n-k-1)(2n-k-2) M_k \gamma^{2n-k}$$

$$a_{24} = -\left[\ln\gamma + \sum_{n=1}^{\infty} \frac{1}{2n} M_0 \gamma^{2n}\right], \qquad a_{64} = -\gamma\Big[-4 + 4a_0 + b\alpha^2(1 + \ln\alpha)$$

$$+ \sum_{n=1}^{\infty} R_0 M_0 \alpha^{2n}\Big]\alpha^{-1} + 4a_0\left[1 + \sum_{n=1}^{\infty} M_0 \gamma^{2n}\right] + \sum_{n=1}^{\infty} (2n-1)(2n-2) M_0 \gamma^{2n} + 2$$

$$R_k = -2 + 4a_0 + b\alpha^2 + b\alpha^2/(2n-k) + (2n-k-1)(2n-k+2)$$

Further, we note that the elements of the second column of the determinant coincide with the corresponding elements of the first column if in these latter we replace the Bessel function of the first kind I_ν by the Bessel function of the second kind Y_ν. The elements of the fifth column may be obtained from the elements of the sixth if in the latter we formally replace k by $-k$; and, finally, the elements of the fourth column may be obtained from the elements of the fifth for $k = 0$, except for the elements a_{24} and a_{64}, which are as shown above. In Fig. 1 we show the dependence of the critical pressure p_0 on α for $0.1 \le \rho_0 \le 1$, $0 \le c_0 \le 1$, $k_0 = 0.1$ and $\tau = 1$. The magnitude of the elastic–plastic boundary γ corresponding to the critical load p_0 for these values of ρ_0, c_0, k_0, and τ is presented in Fig. 2a. We remark that as k_0 decreases the size of the critical pressure decreases, which follows from Eq. (2.3), since $p_0 = k_0\{\ldots\}$; and the formulation of p_0 as a function of α for $0 < k_0 < 1$ is readily made.

The assumption that loss of stability may occur in accord with the type of static instability leads to essential simplifications.

Without making the intermediate calculations, we merely indicate that the equation for determining the critical pressure is obtained upon expanding the determinant (2.22), which in the given case has the following elements:

$$\begin{gathered} a_{11} = \ln\gamma,\ a_{21} = 1,\ a_{31} = -1,\ a_{41} = -2(1+2\gamma),\ a_{51} = a_{61} = 0 \\ a_{12} = \gamma^2,\ a_{22} = a_{32} = 2\gamma^2,\ a_{42} = -12\gamma,\ a_{52} = 0,\ a_{62} = 1 \\ a_{13} = \gamma^{-2},\ a_{23} = -2\gamma^{-2},\ a_{33} = 6\gamma^{-2},\ a_{43} = -12\gamma(1-2\gamma),\ a_{53} = 0,\ a_{63} = 1 \\ a_{14} = -\ln\gamma,\ a_{24} = -1,\ a_{34} = 1,\ a_{44} = 2[1 + 2a_0(1-\gamma\alpha^{-1}) + 2\gamma\alpha^{-1}] \\ a_{54} = a_{64} = 0,\ a_{15} = -\sin(k\ln\gamma),\ a_{25} = -k\cos(k\ln\gamma) \\ a_{35} = k(k\sin k\ln\gamma + \cos k\ln\gamma),\quad a_{45} = 3k^2(\sin k\ln\gamma - \gamma\alpha^{-1}\sin k \\ \cdot\ln\alpha) + k(2 - k^2 + 4a_0)(\cos k\ln\gamma - \gamma\alpha^{-1}\cos k\ln\alpha) + 6k\gamma\alpha^{-1}\cos k\ln\alpha \\ a_{55} = \sin k\ln\alpha,\qquad a_{65} = 0,\ a_{16} = -\cos k\ln\gamma \\ a_{26} = k\sin k\ln\gamma,\ a_{36} = k(k\cos k\ln\gamma - \sin k\ln\gamma) \\ a_{46} = 3k^2(\cos k\ln\gamma - \gamma\alpha^{-1}\cos k\ln\alpha) - k(2 - k^2 + 4a_0)(\sin k\ln\gamma \\ - \gamma\alpha^{-1}\sin k\ln\alpha) - 6k\gamma\alpha^{-1}\sin k\ln\alpha,\quad a_{56} = \cos k\ln\alpha,\quad a_{66} = 0 \end{gathered} \tag{2.24}$$

Consideration of the static problem, corresponding to the system (2.4) with the conservative boundary conditions (1.6), leads to a determinant whose elements coincide with the elements (2.24), except for the elements

$$a_{44} = 2 + 4a_0(1-\gamma\alpha^{-1}) + (4-p_0)\gamma\alpha^{-1}, \qquad a_{54} = -p_0$$

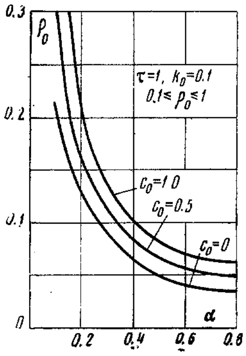

Fig. 1

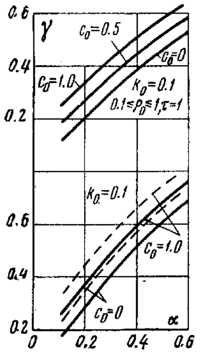

Fig. 2

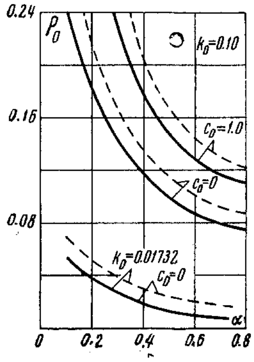

Fig. 3

$$
\begin{aligned}
a_{45} &= 3k^2 (\sin k \ln \gamma - \gamma\alpha^{-1} \sin k \ln \alpha) + k (2 - k^2 + 4a_0) (\cos k \ln \gamma \\
&\quad - \gamma\alpha^{-1} \cos k \ln \alpha) + (6 - p_0) k\gamma\alpha^{-1} \cos k \ln \alpha \\
a_{55} &= (1 + p_0) k^2 \sin k \ln \alpha - p_0 k \cos k \ln \alpha, \quad a_{46} = 3k^2 (\cos k \ln \gamma \\
&\quad - \gamma\alpha^{-1} \cos k \ln \alpha) - k (2 - k^2 + 4a_0) (\sin k \ln \gamma - \gamma\alpha^{-1} \sin k \ln \alpha) \\
&\quad - (6 - p_0) k\gamma\alpha^{-1} \sin k \ln \alpha \\
a_{56} &= (1 + p_0) k^2 \cos k \ln \alpha + p_0 k \sin k \ln \alpha
\end{aligned} \tag{2.25}
$$

Fig. 3 shows the dependence of the critical values of p_0 and γ on α for Eq. (2.22) with the coefficients (2.25) (continuous curves), and for Eq. (2.22) with the elements (2.24) (dashed curves).

As is evident from Fig. 2b and Fig. 3, the magnitude of the critical force, calculated using the conservative boundary conditions, does not differ significantly from that calculated using the nonconservative boundary conditions.

A comparison with [2] (for $k_0 = 0.01732$ and $c_0 = 0$) shows that the magnitude of γ is somewhat less than it is in [2].

From Figs. 1 and 3 it follows that the presence of viscosity for the plastic deformations diminishes the size of the critical force, i.e., viscosity has a destabilizing effect on the tube.

All computations employed in making the graphs were obtained using the M-20 electronic digital computer.

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